$q-$form field and Hodge duality on brane

Yu-Xiao Liu (刘玉孝)

Lanzhou University (兰州大学)

With C.E. Fu, H. Guo and S.L. Zhang [PRD 93 (2016) 064007]

4th International Workshop on Dark Matter, Dark Energy and Matter-Antimatter Asymmetry

December 29-31, 2016, NTHU, Hsinchu, Taiwan

December 31, 2016

Outline

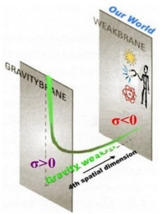
1. Introduction and Motivation
   - p-brane, q-form field, localization
   - Hodge duality in bulk
   - Motivation
2. Localiztion and Hodge duality for $q-$form field on brane
   - Localiztion for $q-$form field on brane
   - Massless KK modes and Hodge duality on the brane
   - Massive KK modes and a new duality
3. Conclusion

Introduction and Motivation

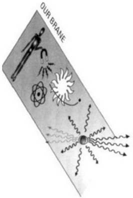
- p-brane, q-form field, localization
- Hodge duality in bulk
- Motivation

♦ $p-$brane (in $D = (p+2)$ dimensions)

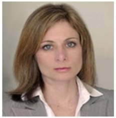
Randall-Sundrum (RS) brnae scenario (Warped extra dimesnion)



RS-I (2 branes)
finite ED

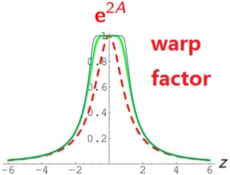


RS-II (1 brane)
infinite ED



Lisa Randall

Raman Sundrum



warp factor

The line element for a Randall-Sundrum brane is

$$ds^2 = e^{2A(y)}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2 = e^{2A(z)}\left(\eta_{\mu\nu}dx^\mu dx^\nu + dz^2\right). \quad (1)$$

♦ Localization of gravity on brane

- Recover GR (Newtonian potential) on brane (massless graviton)
- Correction to Newtonian potential (massive gravitons)

♦ Localization of bulk matter fields on brane

- Realize the Standard model on brane (massless modes)
- Probe extra dimensions (massive KK particles)

♦ $q-$form field $X_{M_1 M_2 \cdots M_q}$

A $q-$form field $X_{M_1 M_2 \cdots M_q}$ is a totally antisymmetric field

$$X_{M_1 M_2 \ldots M_q} = X_{[M_1 M_2 \ldots M_q]}. \qquad (X_{[q]}) \tag{2}$$

Its field strength is expressed as

$$Y_{M_1 M_2 \cdots M_{q+1}} = \partial_{[M_1} X_{M_2 \cdots M_{q+1}]}. \;\; (Y_{[q+1]} = dX_{[q]}) \tag{3}$$

Examples:

- **0-form: Scalar field ϕ**
- **1-form: Vector field A_M, $(-q \int dx^M A_M)$** [1]
- **2-form: Kalb-Ramond field**[2] **B_{MN}, $(-\int dx^M dx^N B_{MN})$** [3]

[1] The action for a charged particle moving in an electromagnetic potential

[2] The KR field (NS-NS B-field) appears, together with the metric tensor and dilaton, as a set of massless excitations of a closed string.

[3] The action for a string coupled to the Kalb-Ramond field. This term in the action implies that the fundamental string of string theory is a source of the NS-NS B-field, much like charged particles are sources of the electromagnetic field.

♦ **Hodge duality in bulk:**

$q-$**form** $\Leftrightarrow (p-q)-$**form in** $p+2$ **dimensions**

A massless $q-$form field is dual to a $(p-q)-$form field:

$$\sqrt{-g}\ \widetilde{Y}^{M_1\cdots M_{p-q+1}} = \frac{1}{(q+1)!}\ \varepsilon^{M_1\cdots M_{p-q+1}N_1\cdots N_{q+1}}\, Y_{N_1\cdots N_{q+1}}. \quad (4)$$

$$S_{bulk,q} = S_{bulk,p-q}. \quad (5)$$

Example: $0-$**form** $\Leftrightarrow 3-$**form** **in 5 dimensions** $(p=3)$

♦ Localization of $q-$form fields on RS brane (Susskind et al)

The bulk action for a massless $q-$form field is

$$S_{bulk,q} = \frac{-1}{2(q+1)!}\int_M Y^{M_1 M_2 \cdots M_{q+1}} Y_{M_1 M_2 \cdots M_{q+1}}, \tag{6}$$

where $\int_M \equiv \int d^D x\sqrt{-g}$. Susskind et al gave the following ansatz [JHEP 0105 (2001) 031] [4]

$$\begin{aligned} X_{\mu_1\cdots\mu_q}(x,y) &= \hat{X}_{\mu_1\cdots\mu_q}(x), \quad \textbf{(zero mode)} \quad &(7a)\\ X_{\mu_1\cdots\mu_{q-1}z}(x,y) &= 0, \quad \textbf{(gauge condition)} \quad &(7b)\end{aligned}$$

so that $Y_{[q+1]} = \hat{Y}_{[q+1]}$. Thus, the brane action reads

$$S_{brane} = \frac{-1}{2(q+1)!}\int_{\partial M} \hat{Y}^{\mu_1\cdots\mu_{q+1}} \hat{Y}_{\mu_1\cdots\mu_{q+1}} \int dy e^{2(q+1-(D-1)/2)k|y|}. \tag{8}$$

[4] N. Kaloper, E. Silverstein, and L. Susskind, JHEP 0105 (2001) 031.

The localization of a $q-$form field on RS brane requires the convergence of the y integral, or

$$q < \frac{D-3}{2} = \frac{p-1}{2}. \tag{9}$$

So the ansatz (7) implies that only 0-forms (scalars) can be localized on the brane in $D = 5$ dimensions.

The above result claims that scalars can be bound to a RS brane, but higher $q-$form fields cannot[5].

This conflicts with the Hodge duality between 0-form and 3-form fields in a 5-dimensional bulk.

[5] N. Kaloper, E. Silverstein, and L. Susskind, Gauge symmetry and localized gravity in M-theory, JHEP 0105 (2001) 031, hep-th/0006192.

♦ Resolution (Duff and Liu)

The resolution of the paradox was given by Duff and Liu[6]:

choose different ansatzs for small and large q

$$X_{\mu_1\cdots\mu_q} = \begin{cases} \hat{X}^{(n)}_{\mu_1\cdots\mu_q}(x), & q < \frac{D-3}{2} \\ \hat{X}^{(n)}_{\mu_1\cdots\mu_q}(x)e^{-(2q-D+1)k|y|}, & q \geq \frac{D-3}{2} \end{cases} \quad (10a)$$

$$X_{\mu_1\cdots\mu_{q-1}z} = 0. \quad (10b)$$

With this choice, Hodge duality on the brane for the massless modes can be kept.

However, the Hodge duality for massive KK modes cannot be obtained.

[6] M. Duff and J.T. Liu, Hodge duality on the brane, PLB 508 (2001) 381

♦ Motivation

Is there other resolution? For which,

- **the KK decomposition of the $q-$form field is suitable for any q,**
- **Hodge duality for the massless modes on the brane is also kept, and more important,**
- **Hodge duality for massive KK modes can be obtained.**

Our resolution:

make a general KK decomposition without gauge choice

$$X_{\mu_1\cdots\mu_q}(x_\mu, z) = \sum_n \hat{X}^{(n)}_{\mu_1\cdots\mu_q}(x^\mu) \quad U^{(n)}_1(z)\mathbf{e}^{a_1 A(z)}, \quad (11a)$$

$$X_{\mu_1\cdots\mu_{q-1}z}(x_\mu, z) = \sum_n \hat{X}^{(n)}_{\mu_1\cdots\mu_{q-1}}(x^\mu)\, U^{(n)}_2(z)\mathbf{e}^{a_2 A(z)}. \quad (11b)$$

where $a_1 = a_2 = (2q-p)/2$.

Localiztion and Hodge duality for $q-$form field on brane

Localiztion for $q-$form field on brane

The action for a massless $q-$form field is

$$S = -\frac{1}{2(q+1)!}\int_M Y^{M_1 M_2 \cdots M_{q+1}} Y_{M_1 M_2 \cdots M_{q+1}}, \tag{12}$$

and the EoMs are

$$\begin{aligned} \partial_{\mu_1}(\sqrt{-g}\; Y^{\mu_1\mu_2\cdots\mu_{q+1}}) + \partial_z(\sqrt{-g}\; Y^{z\mu_2\cdots\mu_{q+1}}) &= 0, \quad (13a)\\ \partial_{\mu_1}(\sqrt{-g}\; Y^{\mu_1\mu_2\cdots\mu_q z}) &= 0, \quad (13b) \end{aligned}$$

Substituting the KK decomposition (11) into the above equaions, we get

$$\frac{1}{\sqrt{-\hat{g}}}\partial_{\mu_1}\left(\sqrt{-\hat{g}}\ \hat{Y}_{(n)}^{\mu_1\mu_2\cdots\mu_{q+1}}\right)+\lambda_1\hat{X}_{(n)}^{\mu_2\cdots\mu_{q+1}}+\lambda_2\hat{Y}_{(n)}^{\mu_2\cdots\mu_{q+1}}=0, \quad (14)$$

$$\partial_{\mu_1}\left(\sqrt{-\hat{g}}\ \hat{Y}_{(n)}^{\mu_1\mu_2\cdots\mu_q}\right)+\lambda_3\partial_{\mu_1}\left(\sqrt{-\hat{g}}\ \hat{X}_{(n)}^{\mu_1\mu_2\cdots\mu_q}\right)=0, \quad (15)$$

where

$$\lambda_1 = \frac{\mathbf{e}^{-(a_1+p-2q)A}}{(q+1)\ U_1^{(n)}}\ \partial_z\left(\mathbf{e}^{(p-2q)A}\ \partial_z\left(U_1^{(n)}\ \mathbf{e}^{a_1A}\right)\right), \quad (16)$$

$$\lambda_2 = \frac{q\ \mathbf{e}^{-(a_1+p-2q)A}}{(q+1)\ U_1^{(n)}}\ \partial_z\left(U_2^{(n)}\ \mathbf{e}^{(a_2+p-2q)A}\right), \quad (17)$$

$$\lambda_3 = \frac{\partial_z(U_1^{(n)}\ \mathbf{e}^{a_1A})}{q\ U_2^{(n)}\ \mathbf{e}^{a_2A}}. \quad (18)$$

Substituting the general KK decomposition (11) into the bulk action for the $q-$form field, we have

$$S_q = \sum_n S_{q,n}, \tag{19}$$

where the effective action for the $n-$level KK modes is

$$\begin{aligned} S_{q,n} &= -\frac{1}{2(q+1)!}\int_{\partial M} \hat{Y}_{(n)}^{\mu_1\cdots\mu_{q+1}}\hat{Y}^{(n)}_{\mu_1\cdots\mu_{q+1}} \\ &\quad -\frac{1}{2q!}\int_{\partial M}\left(\hat{Y}_{(n)}^{\mu_1\cdots\mu_q}+\frac{m_n}{q+1}\hat{X}_{(n)}^{\mu_1\cdots\mu_q}\right)^2. \end{aligned} \tag{20}$$

Here we have assumed that $U_{1,2}^{(n)}(z)$ satisfy the following orthonormality conditions

$$\int dz\, U_1^{(n)}U_1^{(n')} = \delta_{nn'}, \tag{21a}$$

$$\int dz\, U_2^{(n)}U_2^{(n')} = \frac{(q+1)^2}{q^2}\delta_{nn'}. \tag{21b}$$

Equations of motion for $U_1^{(n)}$ and $U_2^{(n)}$ are

$$\mathcal{Q}\mathcal{Q}^\dagger \, U_1^{(n)}(z) = m_n^2 \, U_1^{(n)}(z), \quad (22)$$

$$\mathcal{Q}^\dagger\mathcal{Q} \, U_2^{(n)}(z) = m_n^2 \, U_2^{(n)}(z), \quad (23)$$

with the operator $\mathcal{Q}$ given by $\mathcal{Q} = \partial_z + \frac{p-2q}{2}A'(z)$, which indicates that

- $m_n^2 \geq 0$: **There is no KK modes with negative eigenvalue.**

Solutions of the zero modes $U_{1,2}^{(0)}$ ($m_0 = 0$):

$$U_1^{(0)}(z) = N_1 \, \mathbf{e}^{+(p-2q)A/2}, \quad (24)$$

$$U_2^{(0)}(z) = N_2 \, \mathbf{e}^{-(p-2q)A/2}, \quad (25)$$

Massless KK modes

Their effective action reads

$$S_{q,0} = \int_{\partial M} \left(I_{00}^{(1)} \; \hat{Y}_{(0)}^{\mu_1\cdots\mu_{q+1}} \hat{Y}^{(0)}_{\mu_1\cdots\mu_{q+1}} + I_{00}^{(2)} \; \hat{Y}_{(0)}^{\mu_1\cdots\mu_q} \hat{Y}^{(0)}_{\mu_1\cdots\mu_q} \right), \quad (26)$$

where

$$I_{q,00}^{(1)} = N_1^2 \int dz \, \mathbf{e}^{(p-2q)A}, \quad (27)$$

$$I_{q,00}^{(2)} = N_2^2 \int dz \, \mathbf{e}^{-(p-2q)A}. \quad (28)$$

It is clear that only one of the zero modes can be localized on the RS brane, $q-$form or $(q-1)-$form zero mode.

For the dual $(p-q)-$form field $\widetilde{X}_{\nu_1\nu_2\cdots\nu_{p-q}}$, one has

$$\widetilde{I}^{(1)}_{p-q,00} = \frac{I^{(2)}_{q,00}}{q+1},$$
$$\widetilde{I}^{(2)}_{p-q,00} = (p-q+1)\, I^{(1)}_{q,00}.$$

So,

if there is a localized $q-$form zero mode,
there must be a localized $(p-q-1)-$form one for its dual field.

If there is a localized $(q-1)-$form zero mode,
there must be a localized $(p-q)-$form one.

Hodge duality on the brane for massless modes

Substituting the KK decompositions of the $q-$ and $(p-q)-$form fields into the Hodge duality in the bulk, we obtain the Hodge duality on the brane

$$\sqrt{-\hat{g}}\ \widetilde{\hat{Y}}_{(0)}^{\mu_1\cdots\mu_{p-q}}(x^\mu) = \frac{1}{(q+1)!}\ \varepsilon^{\mu_1\cdots\mu_{p-q}\nu_1\cdots\nu_{q+1}}\ \hat{Y}^{(0)}_{\nu_1\cdots\nu_{q+1}}(x^\mu),$$

$$\sqrt{-\hat{g}}\ \widetilde{\hat{Y}}_{(0)}^{\mu_1\cdots\mu_{p-q+1}}(x^\mu) = \frac{1}{q!}\ \varepsilon^{\mu_1\cdots\mu_{p-q+1}\nu_1\cdots\nu_q}\ \hat{Y}^{(0)}_{\nu_1\cdots\nu_q}(x^\mu).$$

The Hodge duality on the brane just suggests that

- a massless q-form field is dual to a $(p-q-1)$-form one, or
- a massless $(q-1)$-form field to a $(p-q)$-form one.

It can be shown that the corresponding effective actions are the same.

Massive KK modes and a new duality

Massive KK modes
and
a new duality

Massive KK modes and a new duality

The effective action for each $n-$level KK modes of the bulk q-form field is

$$S_{q,n} = -\frac{1}{2(q+1)!}\int_{\partial M} \hat{Y}_{(n)}^{\mu_1\cdots\mu_{q+1}}\hat{Y}^{(n)}_{\mu_1\cdots\mu_{q+1}} - \frac{1}{2q!}\int_{\partial M}\left(\hat{Y}_{(n)}^{\mu_1\cdots\mu_q} + \frac{m_n}{q+1}\hat{X}_{(n)}^{\mu_1\cdots\mu_q}\right)^2,$$

which is for two kinds of KK modes:

- a massive n-level q-form mode with mass m_n and
- a massless n-level $(q-1)$-form mode.

Massive KK modes and a new duality

The effective action for each $n-$level KK modes of the $(p-q)$-form field is

$$\widetilde{S}_{p-q,n} = -\frac{1}{2(p-q+1)!}\int_{\partial M} \widetilde{\hat{Y}}^{\mu_1\cdots\mu_{p-q+1}}_{(n)} \widetilde{\hat{Y}}^{(n)}_{\mu_1\cdots\mu_{p-q+1}} - \frac{1}{2(p-q)!}\int_{\partial M}\left(\widetilde{\hat{Y}}^{\mu_1\cdots\mu_{p-q}}_{(n)} - \frac{m_n}{p-q+1}\widetilde{\hat{X}}^{\mu_1\cdots\mu_{p-q}}_{(n)}\right)^2.$$

The above effective action is also for two kinds of KK modes:

- a massive n-level $(p-q)$-form mode with mass m_n and
- a massless n-level $(p-q-1)$-form mode.

Massive KK modes and a new duality

Substituting the field decompositions into the bulk Hodge duality (4), we obtain the following dual relation on the brane between two groups of $n-$level KK modes:

$$\widetilde{\hat{Y}}^{\mu_1\cdots\mu_{p-q}}_{(n)} - \frac{m_n}{p-q+1}\widetilde{\hat{X}}^{\mu_1\cdots\mu_{p-q}}_{(n)} = \frac{\varepsilon^{\mu_1\cdots\mu_{p-q}\nu_1\cdots\nu_{q+1}}}{(q+1)!\sqrt{-\hat{g}}}\hat{Y}^{(n)}_{\nu_1\cdots\nu_{q+1}}, \tag{30}$$

$$\widetilde{\hat{Y}}^{\mu_1\cdots\mu_{p-q+1}}_{(n)} = \frac{\varepsilon^{\mu_1\cdots\mu_{p-q+1}\nu_1\cdots\nu_q}}{q!\sqrt{-\hat{g}}}\left(\hat{Y}^{(n)}_{\nu_1\cdots\nu_q} + \frac{m_n}{q+1}\hat{X}^{(n)}_{\nu_1\cdots\nu_q}\right). \tag{31}$$

With these relations it can be shown that the following two effective actions are equal:

$$S_{q,n} = \widetilde{S}_{p-q,n} \,. \tag{32}$$

Massive KK modes and a new duality

The duality between two group KK modes:

- **one is an n-level massive q-form KK mode with mass m_n and an n-level massless $(q-1)$-form mode,**
- **another is an n-level $(p-q)$-form mode with the same mass m_n and an n-level massless $(p-q-1)$-form mode.**

Conclusion

Dualities in the bulk and on the brane

		duality
bulk	massless	q-form $\Leftrightarrow$ (p-q)-form
brane	zero mode (n=0)	q-form $\Leftrightarrow$ (p-q-1)-form or (q-1)-form $\Leftrightarrow$ (p-q)-form
	KK modes (n>0)	q-form (m_n) + (q-1)-form (massless) $\Updownarrow$ (p-q)-form (m_n) + (p-q-1)-form (massless)

Thanks for your attention!

Hodge duality on the brane for massless modes

It can be shown that the corresponding effective actions are the same:

$$
\begin{aligned}
S_{q,0} &= -\frac{1}{2(q+1)!}\int_{\partial M} \hat{Y}^{\mu_1\cdots\mu_{q+1}}_{(0)}\hat{Y}^{(0)}_{\mu_1\cdots\mu_{q+1}}, \\
&= \\
\widetilde{S}_{p-q-1,0} &= -\frac{1}{2(p-q)!}\int_{\partial M} \widetilde{\hat{Y}}^{\mu_1\cdots\mu_{p-q}}_{(0)}\widetilde{\hat{Y}}^{(0)}_{\mu_1\cdots\mu_{p-q}},
\end{aligned}
$$

or

$$
\begin{aligned}
S_{q-1,0} &= -\frac{1}{2q!}\int_{\partial M} \hat{Y}^{\mu_1\cdots\mu_{q}}_{(0)}\hat{Y}^{(0)}_{\mu_1\cdots\mu_{q}}, \\
&= \\
\widetilde{S}_{p-q,0} &= -\frac{1}{2(p-q+1)!}\int_{\partial M} \widetilde{\hat{Y}}^{\mu_1\cdots\mu_{p-q+1}}_{(0)}\widetilde{\hat{Y}}^{(0)}_{\mu_1\cdots\mu_{p-q+1}}.
\end{aligned}
$$

Massive KK modes and a new duality

The effective action for each $n-$level KK modes of the bulk q-form field is

$$
\begin{aligned}
S_{q,n} &= -\frac{1}{2(q+1)!}\int_{\partial M} \hat{Y}_{(n)}^{\mu_1\cdots\mu_{q+1}}\hat{Y}^{(n)}_{\mu_1\cdots\mu_{q+1}} \\
&\quad -\frac{1}{2q!}\int_{\partial M}\left(\hat{Y}_{(n)}^{\mu_1\cdots\mu_q}+\frac{m_n}{q+1}\hat{X}_{(n)}^{\mu_1\cdots\mu_q}\right)^2,
\end{aligned}
$$

which is gauge invariant under the following gauge transformation:

$$
\hat{X}^{(n)}_{\mu_1\cdots\mu_q} \rightarrow \hat{X}^{(n)}_{\mu_1\cdots\mu_q}+\partial_{[\mu_1}\hat{\Lambda}_{\mu_2\cdots\mu_q]}, \tag{33}
$$

$$
\hat{X}^{(n)}_{\mu_2\cdots\mu_q} \rightarrow \hat{X}^{(n)}_{\mu_2\cdots\mu_q}-\frac{m_n}{q+1}\hat{\Lambda}_{\mu_2\cdots\mu_q}. \tag{34}
$$

Massive KK modes and a new duality

Then we can fix the gauge by choosing

$$\partial^{\mu_1}\hat{X}^{(n)}_{\mu_1\cdots\mu_q}=0. \tag{35}$$

The above effective action is for two kinds of KK modes:

- a massive n-level q-form mode with mass m_n and
- a massless n-level $(q-1)$-form mode.

Massive KK modes and a new duality

On the other hand, we find that the effective potentials of the $q-$form and its dual $(p-q)-$form fields have the following relationships:

$$\widetilde{V}_{p-q,1}(z)=V_{q,2}(z),\quad \widetilde{V}_{p-q,2}(z)=V_{q,1}(z). \tag{36}$$

And there are some relationships between the KK modes:

$$\widetilde{U}_1^{(n)} = \frac{q}{q+1}U_2^{(n)}, \tag{37a}$$

$$\widetilde{U}_2^{(n)} = \frac{p-q+1}{p-q}U_1^{(n)}. \tag{37b}$$